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SOME CONSTRUCTION OF RECTANGULAR DESIGN WITH VARYING REPLICATES

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Abstract

Some methods of construction of rectangular designs with unequal replications number r_i (i=1, 2) of treatments are described. Such designs are labeled as rectangular designs with varying replicates (RDVR) in literature. In this paper, rectangular designs with two replications r_1 and r_2 are discussed.

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1. Introduction

Sometimes it may not be necessary that all treatments be replicated a constant number of times, it is rather the case in many practical situations that block designs with unequal replications are required. Corsten (1) described some construction and analysis of intra and inter-group balanced designs under the heading balanced block designs with two different numbers of replications say r_1, r_2. The balanced bipartite block designs were studied by Kageyama and Sinha (2), may be considered as an extension of group divisible designs which are two associate class partially balanced incomplete block designs.

The concept of rectangular designs with varying replicates (RDVR) was introduced by Sinha, Das and Kageyama (3), where construction of rectangular design with varying replicates was given for prime numbers. In this paper, we describe some constructions of RDVR's using incidence matrices of BIB designs.

An incomplete block design with a set of v treatments arranged into b blocks each of size k (<v) is called rectangular design with varying replicates (RDVR) if:

(i) the v = mn treatments can be arranged into m rows and n columns;

(ii) each treatment in the i^{th} column is replicated exactly in r_i blocks, (i=1,2,….,n);

(iii) with each treatment in the i^{th} column,(i=1,2,…,n), the other treatments;

(a) belonging to the same row occur together λ_1 times, (that is they are called first associates);

(b) belonging to the same column occur together $\lambda_{i\ (2)}$ times, (that is they are called second associates);

(c) otherwise the occure together λ_3 times, (that is they are called third associates);

Let n_i be the number of i^{th} associates of any treatment θ (say). In particular, n_1 = (n-1), n_2 = (m-1) and n_3 = (n-1)(m-1).

The following parametric relations hold for a rectangular design with varying replicates:

(i) v = mn;

(ii) $\sum_{j=1}^{3} n_j = (v-1)$, (j = 1,2,3)

(iii) $m\sum_{i=1}^{n} r_i = bk$; (i = 1,2,3..., n)

(iv) $n_1\lambda_1 + n_2\lambda_{i(2)} + n_3\lambda_3 = r_i(k-1)$, (1=1,2,…,n)

The definitions of other terms discussed here are from Raghavarao (4).

2. Construction using BIB designs

Some methods of construction of rectangular design with varying replicates are discussed by Sinha, Das and Kageyama (3). We present some methods of construction of rectangular designs with variying replicates using incidence matrices of BIB designs.

Theorem 2.1: The existence of a BIBD (v = 2k, b_1, r_1, k_1, λ_1) and another BIBD (v, b_2, r_2, k_2, λ_2) implies the existence of a RDVR with parameters,
$v^*=2v$, $b^*=b_1+b_2$, $r_1^*=r_1+r_2$, $r_2^*=r_1+b_2-r_2$, $k^*=v$, $\lambda_1^*=r_1$, $\lambda_{1(2)}^*= \lambda_1+ \lambda_2$, $\lambda_{2(2)}^*= \lambda_1+ b_2-2\, r_2+ \lambda_2$, $\lambda_3^*= \lambda_1+ r_2- \lambda_2$; m-v, n=2 …(2.1)

Proof: Let N be an incidence matrix of BIBD (v = 2k, b_1, r_1, k_1, λ_1) and N* be an incidence matrix of another BIBD ($v=b_2=s$, $r_2=k_2=s-1$, $\lambda_2=s-2$) for s>2, then the structure

$$M = \begin{bmatrix} N & N^* \\ N & j - N^* \end{bmatrix}$$

Yields the RDVR having parameters given in (2.1), where j_{sxt} is an s x t matrix whose all elements are unity.

Let the v* = v x2 treatments of the association scheme are arranged in any array given as follows:

1	v+1
2	v+2
3	v+3
.	.
.	.
.	.
v	2v

Since, the elements (treatments) of the first column of the array occur (r_1+s-1) times, hence $r_1^* = r_1+r_2$. Similarly, elements (treatments) of the second column of the array occur (r_1+1) times giving $r_2^* = r_1+b_2- r_2=r_1+1$. The inner product between i^{th} and $(i+v)^{th}$ rows gives the value of λ_1^* i.e. $r_i(i=1,2,\ldots,n)$. The inner product between i^{th} and $(i+(v+1))^{th}$ rows gives the value of λ_3^* which is equal to $\lambda_1+r_2- \lambda_2$. The elements (treatments) of the first column of the association scheme occur together $\lambda_{1(2)}$ times,(that is equal to ($\lambda_1+\lambda_2$). Similarly elements (treatments) of the second column of the association scheme occur together $\lambda_{2(2)}$ times which is equal to $(b_1+b_2)-2(r_1+r_2)+(\lambda_1+ \lambda_2) = \lambda$ (after simplification). The proof is completed.

Example 2.1: A plan of RDVR can be constructed on taking N as an incidence matrix of BIBD (6,10,5,3,2) and N* is an incidence matrix of SBIBD (6,6,5,5,4) in theorem 2.1, with parameters.

$V^*=12,\ b^*=16,\ r_1^*=10,\ r_2^*=6,\ k^*=6,\ \lambda_1^*=5,\ \lambda_{1(2)}^*=6,\ \lambda_{2(2)}^*=2,\ \lambda_3^*=3;\ m=6,\ n=2.$

The blocks are written as columns:

1	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3
2	2	2	2	2	2	3	3	4	5	3	3	4	5	4	4
3	3	3	4	3	4	4	6	5	6	4	5	6	6	5	6
4	4	5	5	7	7	5	7	7	7	5	8	8	8	9	9
5	6	6	6	8	8	6	9	10	11	6	9	10	11	10	10
12	11	10	9	9	10	8	12	11	12	7	11	12	12	11	12

The 6 x 2 rectangular array is expressed as

1	7
2	8
3	9
4	10
5	11
6	12

Theorem 2.2: The existence of BIB designs (v, b_i, r_i, k_i, λ_i), (i=1,2) implies the existence of a RDVR with parameters,

$v^*=2v$, $b^*=b_1+b_2$, $r_1^*=r_1+r_2$, $r_2^*=b_1-r_1+b_2-r_2$, $k^*=v$, $\lambda_1^*=0$, $\lambda_{1(2)}^*=\lambda_1+\lambda_2$, $\lambda_{2(2)}^*=b_1-2r_1+\lambda_1+b_2-2r_2+\lambda_2$, $\lambda_3^*=r_1-\lambda_1+r_2-\lambda_2$; m=v, n=2.

…(2.2)

Proof: Suppose N be an incidence matrix of a SBIBD (4t+3, 4t+3, 2t+1, 2t+1, t) for t ≥ 1, and N* be an incidence matrix of another SBIBD (s, s, s-1, s-1, s-2) for s > 2 then the incidence structure

$$M=\begin{bmatrix} N & N^* \\ j-N & j-N^* \end{bmatrix}$$

Yields the required RDVR with parameters given in (2.2). The proof follows on the same line as in Theorem 2.1.

Example 2.2: When t=1, consider N as an incidence matrix of SBIBD with parameters v=7, $k_1=3$, $\lambda_1=1$, $b_1=7$, $r_1=3$; when s=7, consider N* as an incidence matrix of SBIBD with parameters v=7, $k_2=6$, $\lambda_2=5$, $b_2=7$, $r_2=6$, then Theorem 2.2 generates a RDVR with parameters, $v^*=14$, $b^*=14$, $r_1^*=9$, $r_2^*=5$, $k^*=7$, $\lambda_1^*=0$, $\lambda_{1(2)}^*=6$, $\lambda_{2(2)}^*=2$, $\lambda_3^*=3$; m=7, n=2.

Remark 2.1: The existence of a Hadamard design $v=b_1=4t-1$, $r_1=k_1=2t-1$, $\lambda_1=t-1$ (t ≥ 1) and another symmetrical BIB design $v=b_2=s$, $r_2=k_2=s-1$, $\lambda_2=s-2$, when s ≥ 3, implies the existence of RDVR with parameters, $v^*=2v$, $b^*=b_1+b_2=4t+s-1$, $r_1^*=r_1+r_2=2t+s-2$, $r_2^*=b_1-r_1+b_2-r_2=2t+1$, $k^*=v=4t-1$, $\lambda_1^*=0$, $\lambda_{1(2)}^*=\lambda_1+\lambda_2=t+s-3$, $\lambda_{2(2)}^*=b_1-2r_1+\lambda_1+b_2-2r_2+\lambda_2=t$, $\lambda_3^*=r_1-\lambda_1+r_2-\lambda2=t+1$; m=v, n=2.

Remark 2.2: The existence of a symmetrical BIBD with parameters, $v=t^2+t+1$, $b_1=t^2+t+1$, r1=t+1, k1=t+1, λ1=1 and another symmetrical BIBD with parameters, v=s, r2=s-1, $k_2=s-1$, λ2=s-2 implies the existence of RDVR with parameters,

$v^*=2v$, $b^*=b_1+b_2=t^2+t+s+1$, $r_1^*=r_1+r_2=t+s$, $r_2^*=b_1-r_1+b_2-r_2=t^2+1$, $k^*=v$, $\lambda_1^*=0$, $\lambda_{1(2)}^*=\lambda_1+\lambda_2=t^2+t$, $\lambda_{2(2)}^*=b_1-2r_1+\lambda_1+b_2-2r_2+\lambda_2=t(t-1)$, $\lambda_3^*=r_1-\lambda_1+r_2-\lambda2=t+1$; m=v, n=2.

Remark 2.3: The existence of a BIBD with parameters v=4t+1, b1=2(4t+1), r1=4t, k1=2t, λ1=2t-1, for t $\geq$ 1 and another symmetric BIBD with parameters v=s, b_2=s, r_2=s-1, k_2=s-1, λ_2=s-2 for s $\geq$ 3, implies the existence of RDVR with parameters, v*=2v, b*=b_1+b_2=2(4t+1)+s, r_1*=r_1+r_2=4t+s-1, r_2*=b_1- r_1+b_2-r_2=4t+3, k*=v=4t-1, λ_1*=0, $\lambda_{1(2)}$*= λ_1+ λ_2=2t+s-3, $\lambda_{2(2)}$*= b_1-2r_1+λ_1+ b_2-2 r_2+λ_2=2t+1, λ_3*=r_1- λ_1+ r_2- λ2=2(t+1); m=v, n=2.

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